

Effective string theory and the long-range relativistic corrections to the quark-antiquark potential

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Abstract

The complete expression of the heavy quark-antiquark potential up to order $1/m^2$ is known from QCD in terms of Wilson loop expectation values. We use that expression and a mapping, assumed to be valid at large distances, between Wilson loop expectation values and correlators evaluated in the effective string theory, to compute the potential. We obtain previously unknown results for the spin and momentum-independent parts of the potential. These are linearly rising with the distance and may be interpreted as relativistic corrections to the string tension. We confirm known results for the other parts of the potential. Finally, we compute the discrete spectrum of a heavy quark-antiquark pair whose interaction is just given by the obtained potential.

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I. INTRODUCTION

Wilson loops have been related to the heavy quark-antiquark potential since the inception of QCD [1–7]. This relation has been put in a systematic framework by non-relativistic effective field theories of QCD [8–11]. In this framework, the heavy quark-antiquark potential is organized as an expansion in $1/m$, where m is the generic heavy-quark mass, while non-analytic terms in $1/m$ factorize. Non-analytic terms may be identified with the Wilson coefficients of non-relativistic QCD (NRQCD), which is the effective field theory that follows from QCD by integrating out modes that scale like m [12, 13]. The order $1/m^0$ potential is the static potential. It is related to the expectation value of a rectangular Wilson loop stretching over time and over the distance between the heavy quark and antiquark. Contributions to the potential of higher orders in $1/m$ are expressed in terms of expectation values of chromoelectric and chromomagnetic field insertions on a rectangular Wilson loop. These, as well as the Wilson loop, are gauge invariant. At order $1/m^2$, the potential is momentum and spin dependent.

The heavy quark-antiquark potential is a function of r , the distance between the heavy quark and antiquark, and Λ_{QCD} , the typical hadronic scale. The potential may be evaluated perturbatively for $r\Lambda_{\text{QCD}} \ll 1$, but it cannot be for $r\Lambda_{\text{QCD}} \gtrsim 1$. The situation $r\Lambda_{\text{QCD}} \gtrsim 1$ is particularly relevant for excited charmonium and bottomonium states and for this reason has been extensively studied in lattice QCD [14–23]. The most recent determinations are in [24–28]. However, not all the long-range contributions to the heavy quark-antiquark potential have been computed on the lattice. While the order $1/m^0$ and $1/m$ contributions have been computed, as well as at order $1/m^2$ the spin and momentum-dependent potentials, an evaluation of the spin and momentum-independent $1/m^2$ potentials in the long range is still missing. The reason is that they involve Wilson loops with three or four field insertions, whose lattice determination is difficult.

The static potential measured by (quenched) lattice simulations exhibits a typical Cornell-potential type behaviour with a Coulombic short-range part and a linear-rising long-range tail. In the long range, $r\Lambda_{\text{QCD}} \gg 1$, a linear potential is predicted by the effective string theory (EST) [29]. Long-range corrections to the linear potential have been calculated in the EST and confirmed by lattice simulations [30–33]. In [34] a one-to-one correspondence between correlators of string coordinates and field insertions on a rectangular Wilson loop

was suggested and used to evaluate the spin-spin potential. Following that approach, in [35] the $1/m$ potential as well as all momentum and spin-dependent $1/m^2$ potentials were evaluated in the EST. Remarkably, in all the available cases the long-range behaviour of the (quenched) lattice data agrees with the EST determination.¹ This suggests that the EST may serve to evaluate the long-range behaviour of the still unknown spin and momentum-independent $1/m^2$ potentials, providing at the same time a non-trivial prediction for future lattice determinations and the missing ingredient needed to include all $1/m^2$ potentials in the computation of the quarkonium spectrum. The aim of this work is to address such an evaluation.

The paper is organized in the following way. In section II, we establish our notation and write the heavy quark-antiquark potential in terms of Wilson loop expectation values. In section III, we review the EST. In section IV, we derive the potential up to order $1/m^2$ in terms of EST correlators and in section V we look at the impact of the different parts of the $1/m^2$ potential on the spectrum in a model that includes only the long-range tail of the potential. Finally, in section VI, we draw some conclusions.

II. RELATIVISTIC CORRECTIONS TO THE STATIC POTENTIAL

The complete heavy quark-antiquark potential up to order $1/m^2$ has been written in terms of Wilson loop expectation values in [8, 9]. We will use here the same notations and expressions, which we recall shortly in the next two sections.

A. The structure of the potential

We consider a heavy quark of mass m_1 located at \mathbf{x}_1 and a heavy antiquark of mass m_2 located at \mathbf{x}_2 . The spin and momentum operators of the two particles are respectively $\mathbf{S}_1 \equiv \boldsymbol{\sigma}_1/2$ and $\mathbf{p}_1 \equiv -i\nabla_{\mathbf{x}_1}$, and $\mathbf{S}_2 \equiv \boldsymbol{\sigma}_2/2$ and $\mathbf{p}_2 \equiv -i\nabla_{\mathbf{x}_2}$. The distance between the quark and the antiquark is $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$. Up to order $1/m^2$ the quark-antiquark potential

¹ For the spin and momentum-dependent $1/m^2$ potentials these results were known for a long time in an equivalent approach to the EST that consists in approximating the Wilson loop with the exponential of its rectangular area [7, 36–39]. See also [22].

can be written as the sum of three terms,

$$V = V^{(0)} + V^{(1/m)} + V^{(1/m^2)}, \quad (1)$$

where $V^{(0)}(r)$ is the static potential,

$$V^{(1/m)}(r) = \frac{V^{(1,0)}(r)}{m_1} + \frac{V^{(0,1)}(r)}{m_2}, \quad (2)$$

the $1/m$ potential and

$$V^{(1/m^2)} = \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2}, \quad (3)$$

the $1/m^2$ potential. Invariance under charge conjugation and particle interchange implies $V^{(1,0)}(r) = V^{(0,1)}(r)$. It is useful to separate in the $1/m^2$ potential a spin-dependent (SD) from a spin-independent (SI) part:

$$V^{(2,0)} = V_{SD}^{(2,0)} + V_{SI}^{(2,0)}, \quad (4)$$

$$V^{(0,2)} = V_{SD}^{(0,2)} + V_{SI}^{(0,2)}, \quad (5)$$

where

$$V_{SI}^{(2,0)} = \frac{1}{2} \left\{ \mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r) \right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2} \mathbf{L}_1^2 + V_r^{(2,0)}(r), \quad (6)$$

$$V_{SI}^{(0,2)} = \frac{1}{2} \left\{ \mathbf{p}_2^2, V_{\mathbf{p}^2}^{(0,2)}(r) \right\} + \frac{V_{\mathbf{L}^2}^{(0,2)}(r)}{r^2} \mathbf{L}_2^2 + V_r^{(0,2)}(r), \quad (7)$$

and $\mathbf{L}_i = \mathbf{r} \times \mathbf{p}_i$ with $i = 1, 2$. Also in this case invariance under charge conjugation and particle interchange yields

$$V_{\mathbf{p}^2}^{(2,0)}(r) = V_{\mathbf{p}^2}^{(0,2)}(r), \quad (8)$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = V_{\mathbf{L}^2}^{(0,2)}(r), \quad (9)$$

$$V_r^{(2,0)}(r) = V_r^{(0,2)}(r; m_2 \leftrightarrow m_1). \quad (10)$$

For the spin-dependent part we have

$$V_{SD}^{(2,0)} = V_{LS}^{(2,0)}(r) \mathbf{L}_1 \cdot \mathbf{S}_1, \quad (11)$$

$$V_{SD}^{(0,2)} = -V_{LS}^{(0,2)}(r) \mathbf{L}_2 \cdot \mathbf{S}_2. \quad (12)$$

Charge conjugation and particle interchange invariance imply $V_{LS}^{(2,0)}(r) = V_{LS}^{(0,2)}(r; m_2 \leftrightarrow m_1)$. One proceeds similarly for the $V^{(1,1)}$ potential:

$$V^{(1,1)} = V_{SD}^{(1,1)} + V_{SI}^{(1,1)}, \quad (13)$$

where

$$V_{SI}^{(1,1)} = -\frac{1}{2} \left\{ \mathbf{p}_1 \cdot \mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r) \right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2} (\mathbf{L}_1 \cdot \mathbf{L}_2 + \mathbf{L}_2 \cdot \mathbf{L}_1) + V_r^{(1,1)}(r), \quad (14)$$

and

$$V_{SD}^{(1,1)} = V_{L_1 S_2}^{(1,1)}(r) \mathbf{L}_1 \cdot \mathbf{S}_2 - V_{L_2 S_1}^{(1,1)}(r) \mathbf{L}_2 \cdot \mathbf{S}_1 + V_{S_2^2}^{(1,1)}(r) \mathbf{S}_1 \cdot \mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r) \mathbf{S}_{12}(\hat{\mathbf{r}}), \quad (15)$$

with

$$\mathbf{S}_{12}(\hat{\mathbf{r}}) \equiv 3 \hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_1 \hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (16)$$

and $V_{L_1 S_2}^{(1,1)}(r) = V_{L_2 S_1}^{(1,1)}(r; m_1 \leftrightarrow m_2)$.

B. The potential in QCD

In the following, we list the potentials $V^{(i,j)}(r)$ written in terms of operator insertions on a rectangular Wilson loop. We refer the reader to [8, 9] for the derivation of these expressions and for further details.

The static potential is given by

$$V^{(0)}(r) = \lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle W_{\square} \rangle, \quad (17)$$

where $\langle W_{\square} \rangle$ is the expectation value of the rectangular Wilson loop,

$$W_{\square} \equiv \text{P exp} \left\{ -ig \oint_{r \times T} dz^{\mu} A_{\mu}(z) \right\}, \quad (18)$$

and P stands for the path ordering of the color matrices [4]. We also define $\langle\langle \dots \rangle\rangle \equiv \langle \dots W_{\square} \rangle / \langle W_{\square} \rangle$ and the connected correlators

$$\langle\langle O_1(t_1) O_2(t_2) \rangle\rangle_c = \langle\langle O_1(t_1) O_2(t_2) \rangle\rangle - \langle\langle O_1(t_1) \rangle\rangle \langle\langle O_2(t_2) \rangle\rangle, \quad (19)$$

$$\begin{aligned} \langle\langle O_1(t_1) O_2(t_2) O_3(t_3) \rangle\rangle_c &= \langle\langle O_1(t_1) O_2(t_2) O_3(t_3) \rangle\rangle - \langle\langle O_1(t_1) \rangle\rangle \langle\langle O_2(t_2) O_3(t_3) \rangle\rangle_c \\ &\quad - \langle\langle O_1(t_1) O_2(t_2) \rangle\rangle_c \langle\langle O_3(t_3) \rangle\rangle - \langle\langle O_1(t_1) \rangle\rangle \langle\langle O_2(t_2) \rangle\rangle \langle\langle O_3(t_3) \rangle\rangle, \end{aligned} \quad (20)$$

$$\begin{aligned} \langle\langle O_1(t_1) O_2(t_2) O_3(t_3) O_4(t_4) \rangle\rangle_c &= \langle\langle O_1(t_1) O_2(t_2) O_3(t_3) O_4(t_4) \rangle\rangle \\ &\quad - \langle\langle O_1(t_1) \rangle\rangle \langle\langle O_2(t_2) O_3(t_3) O_4(t_4) \rangle\rangle_c - \langle\langle O_1(t_1) O_2(t_2) \rangle\rangle_c \langle\langle O_3(t_3) O_4(t_4) \rangle\rangle_c \end{aligned}$$

$$\begin{aligned}
& -\langle\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle\rangle_c\langle\langle O_4(t_4)\rangle\rangle - \langle\langle O_1(t_1)\rangle\rangle\langle\langle O_2(t_2)\rangle\rangle\langle\langle O_3(t_3)O_4(t_4)\rangle\rangle_c \\
& -\langle\langle O_1(t_1)\rangle\rangle\langle\langle O_2(t_2)O_3(t_3)\rangle\rangle_c\langle\langle O_4(t_4)\rangle\rangle - \langle\langle O_1(t_1)O_2(t_2)\rangle\rangle_c\langle\langle O_3(t_3)\rangle\rangle\langle\langle O_4(t_4)\rangle\rangle \\
& -\langle\langle O_1(t_1)\rangle\rangle\langle\langle O_2(t_2)\rangle\rangle\langle\langle O_3(t_3)\rangle\rangle\langle\langle O_4(t_4)\rangle\rangle,
\end{aligned} \tag{21}$$

where $O_1(t_1)$, $O_2(t_2)$, ..., $O_n(t_n)$ are operators inserted on the Wilson loop at times $t_1 \geq t_2 \geq \dots \geq t_{n-1} \geq t_n$. Connected correlators are made of Feynman diagrams that cannot be disconnected by cutting once the heavy-quark and antiquark lines.

The $1/m$ potential is given by

$$V^{(1,0)}(r) = -\frac{1}{2} \int_0^\infty dt t \langle\langle g\mathbf{E}_1(t) \cdot g\mathbf{E}_1(0) \rangle\rangle_c, \tag{22}$$

where $\mathbf{E}_i(t)$ (and later $\mathbf{B}_i(t)$) stands for $\mathbf{E}(t, \mathbf{x}_i)$ ($\mathbf{B}(t, \mathbf{x}_i)$) with $i = 1, 2$. The $1/m^2$ potentials are²

$$V_{\mathbf{p}^2}^{(2,0)}(r) = \frac{i}{2} \hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \int_0^\infty dt t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0) \rangle\rangle_c, \tag{23}$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = \frac{i}{4} (\delta^{ij} - 3\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j) \int_0^\infty dt t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0) \rangle\rangle_c, \tag{24}$$

$$V_{LS}^{(2,0)}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r} \cdot \int_0^\infty dt t \langle\langle g\mathbf{B}_1(t) \times g\mathbf{E}_1(0) \rangle\rangle + \frac{c_S^{(1)}}{2r^2} \mathbf{r} \cdot (\nabla_r V^{(0)}), \tag{25}$$

$$V_{\mathbf{p}^2}^{(1,1)}(r) = i\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \int_0^\infty dt t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0) \rangle\rangle_c, \tag{26}$$

$$V_{\mathbf{L}^2}^{(1,1)}(r) = \frac{i}{2} (\delta^{ij} - 3\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j) \int_0^\infty dt t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0) \rangle\rangle_c, \tag{27}$$

$$V_{L_2 S_1}^{(1,1)}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r} \cdot \int_0^\infty dt t \langle\langle g\mathbf{B}_1(t) \times g\mathbf{E}_2(0) \rangle\rangle, \tag{28}$$

$$V_{S^2}^{(1,1)}(r) = \frac{2c_F^{(1)}c_F^{(2)}}{3} i \int_0^\infty dt \langle\langle g\mathbf{B}_1(t) \cdot g\mathbf{B}_2(0) \rangle\rangle - 4(d_{sv} + d_{vv}C_f) \delta^{(3)}(\mathbf{r}), \tag{29}$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{c_F^{(1)}c_F^{(2)}}{4} i\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \int_0^\infty dt \left[\langle\langle g\mathbf{B}_1^i(t) g\mathbf{B}_2^j(0) \rangle\rangle - \frac{\delta^{ij}}{3} \langle\langle g\mathbf{B}_1(t) \cdot g\mathbf{B}_2(0) \rangle\rangle \right], \tag{30}$$

$$\begin{aligned}
V_r^{(2,0)}(r) &= \frac{\pi C_f \alpha_s c_D^{(1)'}}{2} \delta^{(3)}(\mathbf{r}) \\
&\quad - \frac{ic_F^{(1)2}}{4} \int_0^\infty dt \langle\langle g\mathbf{B}_1(t) \cdot g\mathbf{B}_1(0) \rangle\rangle_c + \frac{1}{2} \left(\nabla_r^2 V_{\mathbf{p}^2}^{(2,0)} \right) \\
&\quad - \frac{i}{2} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \langle\langle g\mathbf{E}_1(t_1) \cdot g\mathbf{E}_1(t_2) g\mathbf{E}_1(t_3) \cdot g\mathbf{E}_1(0) \rangle\rangle_c
\end{aligned}$$

² We have dropped terms proportional to $\nabla_r^i V^{(0)}$ in the expressions of $V_r^{(2,0)}(r)$ and $V_r^{(1,1)}(r)$ because they are suppressed in the non-relativistic power counting (see section VI of [9]).

$$\begin{aligned}
& + \frac{1}{2} \left(\nabla_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_1(t_2) \cdot g\mathbf{E}_1(0) \rangle\rangle_c \right) \\
& - d_3^{(1)'} f_{abc} \int d^3\mathbf{x} \lim_{T \rightarrow \infty} g \langle\langle F_{\mu\nu}^a(x) F_{\mu\alpha}^b(x) F_{\nu\alpha}^c(x) \rangle\rangle, \tag{31}
\end{aligned}$$

$$\begin{aligned}
V_r^{(1,1)}(r) = & -\frac{1}{2} \left(\nabla_r^2 V_{\mathbf{p}^2}^{(1,1)} \right) \\
& - i \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \langle\langle g\mathbf{E}_1(t_1) \cdot g\mathbf{E}_1(t_2) g\mathbf{E}_2(t_3) \cdot g\mathbf{E}_2(0) \rangle\rangle_c \\
& + \frac{1}{2} \left(\nabla_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_2(t_2) \cdot g\mathbf{E}_2(0) \rangle\rangle_c \right) \\
& + \frac{1}{2} \left(\nabla_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\langle g\mathbf{E}_2^i(t_1) g\mathbf{E}_1(t_2) \cdot g\mathbf{E}_1(0) \rangle\rangle_c \right) \\
& + (d_{ss} + d_{vs} C_f) \delta^{(3)}(\mathbf{r}). \tag{32}
\end{aligned}$$

The coefficients $c_F^{(i)} = 1 + \mathcal{O}(\alpha_s)$, $c_S^{(i)} = 2c_F^{(i)} - 1$, $c_D^{(i)'} = 1 + \mathcal{O}(\alpha_s)$, $d_3^{(1)'} = \alpha_s/(720\pi) + \mathcal{O}(\alpha_s^2)$ [40], and d_{sv} , d_{vv} , d_{ss} , d_{vs} , which are such that $(d_{sv} + d_{vv} C_f) = \mathcal{O}(\alpha_s^2)$ and $(d_{ss} + d_{vs} C_f) = \mathcal{O}(\alpha_s^2)$ [41], are Wilson coefficients of NRQCD. The natural scale of α_s in these coefficients is of the order of the heavy-quark mass, hence we may expect α_s to be a fairly small number. The constant C_f is the Casimir of the fundamental representation of SU(3): $C_f = 4/3$.

III. THE EFFECTIVE STRING THEORY

The effective string theory hypothesis states that in pure gluodynamics and in the long-distance regime, $r\Lambda_{\text{QCD}} \gg 1$, the expectation value of the rectangular Wilson loop can be given in terms of a string action:

$$\lim_{T \rightarrow \infty} \langle W_\square \rangle = Z \int \mathcal{D}\xi^1 \mathcal{D}\xi^2 e^{iS_{\text{string}}(\xi^1, \xi^2)}, \tag{33}$$

where Z is a constant.³ The string action, S_{string} , can be expanded in a series whose terms involve an increasing number of derivatives acting on the transverse string coordinates $\xi^l = \xi^l(t, z)$ ($l = 1, 2$) [31]. The coordinates ξ^l count like $1/\Lambda_{\text{QCD}}$, whereas derivatives in t and z

³ For a general discussion about our current understanding of the QCD vacuum as it is obtained from lattice gauge theory and the duality to string theory we refer to [42]. For recent developments on the effective theory of long strings we refer to [43, 44]. The effective string theory may also provide a long-distance description for other models, an example being the Abrikosov-Nielsen-Olesen vortices of the abelian Higgs model [45, 46].

acting on them count like $1/r$. Hence, terms in S_{string} with more derivatives are suppressed in the long range by powers of $1/(r\Lambda_{\text{QCD}})$ with respect to terms with less derivatives. Up to terms with only two derivatives, the string action reads

$$S_{\text{string}} = -\sigma \int dt dz \left(1 - \frac{1}{2} \partial_\mu \xi^l \partial^\mu \xi^l \right). \quad (34)$$

Studies constraining the form of the higher-order terms, also by Lorentz invariance, are in [43, 44, 47]. The first next terms in the expansion turn out to involve at least four derivatives and are suppressed by $1/(r\Lambda_{\text{QCD}})^2$ with respect to the kinetic term in (34). Such terms and subleading ones do not affect the results presented in this work and will be neglected in the rest of the paper. Since the string has fixed ends at $z = -r/2$ and $z = r/2$, the transverse coordinates ξ^l satisfy the boundary conditions $\xi^l(t, -r/2) = \xi^l(t, r/2) = 0$. The constant σ , which is of order Λ_{QCD}^2 , is the string tension. Its numerical value is known from lattice QCD determinations. From (17), (33) and (34) it follows that [30, 31]

$$V^{(0)}(r) = \sigma r + \mu - \frac{\pi}{12r} \approx \sigma r, \quad (35)$$

where μ is an unknown regularization-dependent constant and the term $-\pi/(12r)$ is a universal quantum correction known as the Lüscher term.⁴ The last approximation holds in the large distance limit when the Lüscher term may be neglected.

In [34] it was proposed that the mapping (33) could be extended to relate Wilson loops with field strength tensor insertions to correlators of the string fields ξ^l . This would allow to compute in the EST the long-range tail of the potentials listed in section II B: a program started with [34] and expanded in [35]. We will follow this latter reference. Requiring the same symmetry properties for the transverse string coordinates and the operators inserted in the Wilson loop, the following mapping between expectation values of operators inserted in the Wilson loop and EST correlators can be established for $r\Lambda_{\text{QCD}} \gg 1$:

$$\begin{aligned} \langle\langle \dots \mathbf{E}_1^l(t) \dots \rangle\rangle &= \langle \dots \Lambda^2 \partial_z \xi^l(t, r/2) \dots \rangle, \\ \langle\langle \dots \mathbf{E}_2^l(t) \dots \rangle\rangle &= \langle \dots \Lambda^2 \partial_z \xi^l(t, -r/2) \dots \rangle, \\ \langle\langle \dots \mathbf{B}_1^l(t) \dots \rangle\rangle &= \langle \dots \Lambda' \epsilon^{lm} \partial_t \partial_z \xi^m(t, r/2) \dots \rangle, \\ \langle\langle \dots \mathbf{B}_2^l(t) \dots \rangle\rangle &= \langle \dots - \Lambda' \epsilon^{lm} \partial_t \partial_z \xi^m(t, -r/2) \dots \rangle, \end{aligned}$$

⁴ The Lüscher term does depend on the dimension of space-time. In d dimensions it reads $-\pi(d-2)/(24r)$. Equation (35) holds for $d = 4$.

$$\begin{aligned}
\langle\langle \dots \mathbf{E}_1^3(t) \dots \rangle\rangle &= \langle \dots \Lambda''^2 \dots \rangle, \\
\langle\langle \dots \mathbf{E}_2^3(t) \dots \rangle\rangle &= \langle \dots \Lambda''^2 \dots \rangle, \\
\langle\langle \dots \mathbf{B}_1^3(t) \dots \rangle\rangle &= \langle \dots \Lambda''' \epsilon^{lm} \partial_t \partial_z \xi^l(t, r/2) \partial_z \xi^m(t, r/2) \dots \rangle, \\
\langle\langle \dots \mathbf{B}_2^3(t) \dots \rangle\rangle &= \langle \dots - \Lambda''' \epsilon^{lm} \partial_t \partial_z \xi^l(t, -r/2) \partial_z \xi^m(t, -r/2) \dots \rangle,
\end{aligned} \tag{36}$$

where the indices l and m label the transverse coordinates: $l, m = 1, 2$. The tensor ϵ^{lm} is such that $\epsilon^{12} = 1$ and $\epsilon^{lm} = -\epsilon^{ml}$. In the Wilson-loop part of the mapping the heavy quark is located at $\mathbf{x}_1 = (0, 0, r/2)$ and the heavy antiquark at $\mathbf{x}_2 = (0, 0, -r/2)$, which implies $\mathbf{r} = (0, 0, r)$. The constants Λ , Λ' , Λ'' and Λ''' are unknown constants of mass dimension one and of order Λ_{QCD} . The mapping (36) is valid up to corrections that are subleading in the long range in the EST counting. For the purpose of the computation in this paper we will assume the mapping to be exact and neglect subleading corrections. We will comment on the impact of subleading corrections at the end of the next section and in the conclusions.

The right-hand side of (36) is made of correlators of string coordinates ξ^l . The functional integral over the string coordinates is Gaussian (see the string action (34)). So we have that correlators of more than two string fields ξ^l break up into products of two-field correlators and derivatives of them, and that two-field correlators are given by [35]

$$\langle \xi^l(it, z) \xi^m(it', z') \rangle = \frac{\delta^{lm}}{4\pi\sigma} \ln \left(\frac{\cosh[(t-t')\pi/r] + \cos[(z+z')\pi/r]}{\cosh[(t-t')\pi/r] - \cos[(z-z')\pi/r]} \right). \tag{37}$$

The calculation of the different possible right-hand sides of (36) through (37) leads to the EST long-range estimate of the potentials listed in section II B.

IV. THE LONG-RANGE POTENTIAL IN THE EFFECTIVE STRING THEORY

The mapping (36) allows us to evaluate in the long range the Wilson loop expectation values that appear in section II B. Correlators of two string fields are given in (37). Derivatives of two field correlators follow from it straightforwardly. Correlators involving more than two string fields, which come from mapping Wilson loops with \mathbf{B}^3 fields or more than two chromoelectric field insertions, decompose into the product of two string field correlators due to the Gaussian string action. Gaussianity also implies that correlators with an odd number of string fields vanish. Hence the Wilson loop expectation values of section II B map

for $r\Lambda_{\text{QCD}} \gg 1$ into the following expressions:

$$\langle\langle \mathbf{E}_1^i(it) \mathbf{E}_1^j(0) \rangle\rangle_c = \tilde{\delta}^{ij} \frac{\pi \Lambda^4}{4\sigma r^2} \sinh^{-2} \left(\frac{\pi t}{2r} \right), \quad (38)$$

$$\langle\langle \mathbf{E}_1^i(it) \mathbf{E}_2^j(0) \rangle\rangle_c = -\tilde{\delta}^{ij} \frac{\pi \Lambda^4}{4\sigma r^2} \cosh^{-2} \left(\frac{\pi t}{2r} \right), \quad (39)$$

$$\mathbf{r} \cdot \langle\langle \mathbf{B}_1(it) \times \mathbf{E}_1(0) \rangle\rangle = \frac{i\pi^2 \Lambda^2 \Lambda'}{2\sigma r^2} \cosh \left(\frac{\pi t}{2r} \right) \sinh^{-3} \left(\frac{\pi t}{2r} \right), \quad (40)$$

$$\mathbf{r} \cdot \langle\langle \mathbf{B}_1(it) \times \mathbf{E}_2(0) \rangle\rangle = -\frac{i\pi^2 \Lambda^2 \Lambda'}{2\sigma r^2} \sinh \left(\frac{\pi t}{2r} \right) \cosh^{-3} \left(\frac{\pi t}{2r} \right), \quad (41)$$

$$\sum_{l=1}^2 \langle\langle \mathbf{B}_1^l(it) \mathbf{B}_1^l(0) \rangle\rangle_c = \frac{\pi^3 \Lambda'^2}{4\sigma r^4} \sinh^{-4} \left(\frac{\pi t}{2r} \right) \left[2 + \cosh \left(\frac{\pi t}{r} \right) \right], \quad (42)$$

$$\sum_{l=1}^2 \langle\langle \mathbf{B}_1^l(it) \mathbf{B}_2^l(0) \rangle\rangle_c = -\frac{\pi^3 \Lambda'^2}{4\sigma r^4} \cosh^{-4} \left(\frac{\pi t}{2r} \right) \left[2 - \cosh \left(\frac{\pi t}{r} \right) \right], \quad (43)$$

$$\langle\langle \mathbf{B}_1^3(it) \mathbf{B}_1^3(0) \rangle\rangle_c = \frac{\pi^4 \Lambda'''^2}{16\sigma^2 r^6} \sinh^{-6} \left(\frac{\pi t}{2r} \right), \quad (44)$$

$$\langle\langle \mathbf{B}_1^3(it) \mathbf{B}_2^3(0) \rangle\rangle_c = \frac{\pi^4 \Lambda'''^2}{16\sigma^2 r^6} \cosh^{-6} \left(\frac{\pi t}{2r} \right), \quad (45)$$

$$\begin{aligned} \langle\langle \mathbf{E}_1(it_1) \cdot \mathbf{E}_1(it_2) \mathbf{E}_1(it_3) \cdot \mathbf{E}_1(0) \rangle\rangle_c &= \frac{\pi^2 \Lambda^8}{8\sigma^2 r^4} \left[\sinh^{-2} \left(\frac{\pi t_2}{2r} \right) \sinh^{-2} \left(\frac{\pi(t_1 - t_3)}{2r} \right) \right. \\ &\quad \left. + \sinh^{-2} \left(\frac{\pi t_1}{2r} \right) \sinh^{-2} \left(\frac{\pi(t_2 - t_3)}{2r} \right) \right], \quad (46) \end{aligned}$$

$$\begin{aligned} \langle\langle \mathbf{E}_1(it_1) \cdot \mathbf{E}_1(it_2) \mathbf{E}_2(it_3) \cdot \mathbf{E}_2(0) \rangle\rangle_c &= \frac{\pi^2 \Lambda^8}{8\sigma^2 r^4} \left[\cosh^{-2} \left(\frac{\pi t_2}{2r} \right) \cosh^{-2} \left(\frac{\pi(t_1 - t_3)}{2r} \right) \right. \\ &\quad \left. + \cosh^{-2} \left(\frac{\pi t_1}{2r} \right) \cosh^{-2} \left(\frac{\pi(t_2 - t_3)}{2r} \right) \right], \quad (47) \end{aligned}$$

where $\tilde{\delta}^{ij} = 0$ for i or $j = 3$ and $\tilde{\delta}^{ij} = \delta^{ij}$ for $i, j = 1, 2$. The expressions for the Wilson loop expectation values with two chromomagnetic or chromoelectric field insertions agree with those in [34]. Terms of the type $\langle\langle \mathbf{E}^i(t_1) \mathbf{E}(t_2) \cdot \mathbf{E}(0) \rangle\rangle_c$ vanish after (36) regardless of the quark line where the chromoelectric fields are located. This is due to Gaussianity and to the subtraction of the disconnected parts; see (20).⁵ Terms involving four chromoelectric fields contribute in the EST through diagrams made of two two-field correlators that are connected.

⁵ It is also a specific feature of $\langle\langle \mathbf{E}^i(t_1) \mathbf{E}(t_2) \cdot \mathbf{E}(0) \rangle\rangle_c$, which is the only type of three-field correlator appearing in the heavy quark-antiquark potential up to order $1/m^2$. For example, a term like $\langle\langle \mathbf{E}^j(t_1) \mathbf{E}^3(t_2) \mathbf{E}^j(0) \rangle\rangle_c$ would not vanish after (36).

Substituting (38)-(47) in the expressions of the potentials, we obtain

$$V^{(1,0)}(r) = \frac{g^2 \Lambda^4}{2\pi\sigma} \ln(\sigma r^2) + \mu_1, \quad (48)$$

$$V_{\mathbf{p}^2}^{(2,0)}(r) = 0, \quad (49)$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = -\frac{g^2 \Lambda^4 r}{6\sigma}, \quad (50)$$

$$V_{LS}^{(2,0)}(r) = -\frac{\mu_2}{r} - \frac{c_F^{(1)} g^2 \Lambda^2 \Lambda'}{\sigma r^2}, \quad (51)$$

$$V_{\mathbf{p}^2}^{(1,1)}(r) = 0, \quad (52)$$

$$V_{\mathbf{L}^2}^{(1,1)}(r) = \frac{g^2 \Lambda^4 r}{6\sigma}, \quad (53)$$

$$V_{L_2 S_1}^{(1,1)}(r) = -\frac{c_F^{(1)} g^2 \Lambda^2 \Lambda'}{\sigma r^2}, \quad (54)$$

$$V_{S^2}^{(1,1)}(r) = \frac{2\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{45\sigma^2 r^5} - 4(d_{sv} + d_{vv} C_f) \delta^{(3)}(\mathbf{r}), \quad (55)$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{90\sigma^2 r^5}, \quad (56)$$

$$V_r^{(2,0)}(r) = -\frac{2\zeta_3 g^4 \Lambda^8 r}{\pi^3 \sigma^2} + \mu_3 + \frac{\mu_4}{r^2} + \frac{\mu_5}{r^4} + \frac{\pi^3 c_F^{(1)2} g^2 \Lambda'''^2}{60\sigma^2 r^5} \\ + \frac{\pi C_f \alpha_s c_D^{(1)'}}{2} \delta^{(3)}(\mathbf{r}) - d_3^{(1)'} f_{abc} \int d^3 \mathbf{x} \lim_{T \rightarrow \infty} g \langle\langle F_{\mu\nu}^a(x) F_{\mu\alpha}^b(x) F_{\nu\alpha}^c(x) \rangle\rangle, \quad (57)$$

$$V_r^{(1,1)}(r) = -\frac{\zeta_3 g^4 \Lambda^8 r}{2\pi^3 \sigma^2} + (d_{ss} + d_{vs} C_f) \delta^{(3)}(\mathbf{r}), \quad (58)$$

where $\zeta_3 = 1.2020569\dots$ is the Riemann zeta function of argument three⁶ and μ_i are renormalization constants. The expressions for the potentials (48)-(54) agree with those in [35]. The spin-spin potentials (55) and (56) are of order $1/r^5$. The $1/r^5$ behaviour comes from the subleading correlator (45), for the long-range leading contribution coming from the correlator (43), which would be of order $1/r^3$, vanishes in the integrals of (29) and (30) (the result is independent on the specific form of the string action). This contrasts with the result of [34], where the correlator (45) is not taken into account and the leading spin-spin

⁶ It comes from the integrals

$$\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 [\sinh^{-2} t_2 \sinh^{-2}(t_1 - t_3) + \sinh^{-2} t_1 \sinh^{-2}(t_2 - t_3)] = \\ 8 \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 [\cosh^{-2} t_2 \cosh^{-2}(t_1 - t_3) + \cosh^{-2} t_1 \cosh^{-2}(t_2 - t_3)] = \zeta_3.$$

potentials shows up only at order $1/m^4$.⁷ The explicit expressions (55) and (56) are new. The potentials (57) and (58) are also new. We observe that correlators of two chromoelectric fields contracted with $\mathbf{r} = (0, 0, r)$ vanish because of $\mathbf{r}^i \tilde{\delta}^{ij} = 0$, and that we do not have a mapping prescription into the EST for the matrix element $\langle\langle F_{\mu\nu}^a(x) F_{\mu\alpha}^b(x) F_{\nu\alpha}^c(x) \rangle\rangle$ involving three gluon fields located at an arbitrary point x of space-time. The expressions listed here correct some of the preliminary findings reported in [48].

As pointed out in [35], Poincaré invariance fixes some of the renormalization constants μ_i and field normalization constants, Λ , Λ' , ..., because it requires some equations to be exactly fulfilled by the potentials (see [49, 50]). One of these equations is the Gromes relation that relates the spin-orbit potentials with the static potential [51]:

$$\frac{1}{2r} \frac{dV^{(0)}}{dr} + V_{LS}^{(2,0)} - V_{L_2 S_1}^{(1,1)} = 0. \quad (59)$$

This equation is fulfilled in the EST only if

$$\mu_2 = \frac{\sigma}{2}. \quad (60)$$

Another equation relates the momentum-dependent potentials with the static potential [36]:⁸

$$\frac{r}{2} \frac{dV^{(0)}}{dr} + 2V_{\mathbf{L}^2}^{(2,0)} - V_{\mathbf{L}^2}^{(1,1)} = 0. \quad (61)$$

This equation is fulfilled in the EST only if

$$g\Lambda^2 = \sigma. \quad (62)$$

A similar relation holds for Λ'' and follows from the equation $-\nabla_1 V^{(0)} = \langle\langle g\mathbf{E}_1 \rangle\rangle$ valid for $T \rightarrow \infty$ derived in [8]. The equation is fulfilled in the EST only if

$$g\Lambda''^2 = -\sigma. \quad (63)$$

⁷ The behaviour of the spin-spin potentials and the disagreement with [34] has been pointed out in [35]. We thank Joan Soto for addressing our attention to this point.

⁸ In [36, 49] also the exact relation

$$-4V_{\mathbf{p}^2}^{(2,0)} + 2V_{\mathbf{p}^2}^{(1,1)} - V^{(0)} + r \frac{dV^{(0)}}{dr} = 0,$$

was derived. This relation is automatically fulfilled by the potentials (35), (49) and (52) in the long range, i.e., neglecting μ and the Lüscher term in $V^{(0)}$, and does not provide further constraints.

Equations (62) and (63) are remarkable, for they completely determine the long-range mapping of the chromoelectric field in the EST. Finally, we note that the equations induced by Poincaré invariance would require the inclusion of subleading corrections to the action (34) and the mapping (36) in order to be fulfilled beyond leading order in the long-range limit.

Taking the potentials (48)-(58) at leading order in the long-range limit, using the constraints (60) and (62), and dropping terms suppressed by powers of α_s , like the term proportional to $\langle\langle F_{\mu\nu}^a(x) F_{\mu\alpha}^b(x) F_{\nu\alpha}^c(x) \rangle\rangle$, we obtain

$$V^{(1,0)}(r) = \frac{\sigma}{2\pi} \ln(\sigma r^2) + \mu_1, \quad (64)$$

$$V_{\mathbf{p}^2}^{(2,0)}(r) = 0, \quad (65)$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = -\frac{\sigma r}{6}, \quad (66)$$

$$V_{LS}^{(2,0)}(r) = -\frac{\sigma}{2r} - \frac{c_F^{(1)} g \Lambda'}{r^2}, \quad (67)$$

$$V_{\mathbf{p}^2}^{(1,1)}(r) = 0, \quad (68)$$

$$V_{\mathbf{L}^2}^{(1,1)}(r) = \frac{\sigma r}{6}, \quad (69)$$

$$V_{L_2 S_1}^{(1,1)}(r) = -\frac{c_F^{(1)} g \Lambda'}{r^2}, \quad (70)$$

$$V_{S^2}^{(1,1)}(r) = \frac{2\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'^2}{45\sigma^2 r^5}, \quad (71)$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'^2}{90\sigma^2 r^5}, \quad (72)$$

$$V_r^{(2,0)}(r) = -\frac{2\zeta_3 \sigma^2 r}{\pi^3}, \quad (73)$$

$$V_r^{(1,1)}(r) = -\frac{\zeta_3 \sigma^2 r}{2\pi^3}. \quad (74)$$

We have kept the subleading term proportional to $1/r^2$ in (67), because (35) and (70) together with (59) guarantee that there cannot be any other term proportional to $1/r^2$ contributing to $V_{LS}^{(2,0)}$. Equations (64)-(74) provide the EST expressions for the heavy quark-antiquark potential in the long range following from the exact mapping (36). Power-counting arguments imply that subleading corrections to the mapping will not change the functional dependence of the potential but may affect some of the numerical coefficients. This can be the case for the spin-spin potentials, which at order $1/r^5$ may be affected by subleading contributions proportional to two string fields in the mapping of \mathbf{B}^l , and for the potentials $V_r^{(2,0)}(r)$ and $V_r^{(1,1)}(r)$, which at order r may be affected by subleading contributions proportional to two string fields in the mapping of \mathbf{E}^3 . In this last case, we note that all terms

proportional to $\Lambda''^8 r^5$, $\Lambda''^6 r^3$ and $\Lambda''^4 (\Lambda^4/\sigma) r^3$ vanish after subtraction of the disconnected parts of the correlators.

V. SPECTRUM

In order to illustrate the impact on the spectrum of the new long-range potentials derived in the previous section, we consider the following model: a quark-antiquark pair both of mass m bound by the potential given in (64)-(74). In the centre-of-mass frame, the Hamiltonian of the system is $H = \mathbf{p}^2/m + V$. The potential, V , reads

$$\begin{aligned}
V(r) = & V^{(0)}(r) + \frac{2}{m}V^{(1,0)}(r) + \frac{1}{m^2} \left\{ \left[2\frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2} + \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{r^2} \right] \mathbf{L}^2 \right. \\
& + \left[V_{LS}^{(2,0)}(r) + V_{L_2 S_1}^{(1,1)}(r) \right] \mathbf{L} \cdot \mathbf{S} + V_{S^2}^{(1,1)}(r) \left(\frac{\mathbf{S}^2}{2} - \frac{3}{4} \right) + V_{\mathbf{S}_{12}}^{(1,1)}(r) \mathbf{S}_{12}(\hat{\mathbf{r}}) \\
& \left. + 2V_r^{(2,0)}(r) + V_r^{(1,1)}(r) \right\} \\
\approx & \sigma r + \frac{1}{m} \frac{\sigma}{\pi} \ln(\sigma r^2) + \frac{1}{m^2} \left(-\frac{\sigma}{6r} \mathbf{L}^2 - \frac{\sigma}{2r} \mathbf{L} \cdot \mathbf{S} - \frac{9\zeta_3 \sigma^2 r}{2\pi^3} \right), \tag{75}
\end{aligned}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and \mathbf{S} is the total spin of the system. In the last line we have dropped contributions to the static and spin-orbit potentials that are subleading in the long range, and the spin-spin potentials, which fall off sharply like $1/r^5$. The constants in the static and $1/m$ potentials do not contribute to the energy level splittings; hence we do not display them. The model has the advantage of depending only on two parameters: the mass m and the string tension σ .

We compute the energy levels by including contributions from the potential that are first order in $1/m^2$ and up to second order in $1/m$. We call $E_{nl}^{(0)}$ the eigenvalues of the zeroth-order Hamiltonian $\mathbf{p}^2/m + \sigma r$. The eigenstates of the zeroth-order Hamiltonian, $|nljs\rangle$, may be chosen to be simultaneously eigenstates of the angular momenta and spin. They are labeled by n, l, j and s , which are the principal, orbital angular momentum, total angular momentum and spin quantum numbers. The state $|nl\rangle$ stands for $|nljs\rangle$ when acting on an

operator that does not depend on spin. The energy levels read⁹

$$E_{nljs} = E_{nl}^{(0)} + \langle nl|V^{(1/m)}|nl\rangle + \sum_{(n',l') \neq (n,l)} \frac{|\langle nl|V^{(1/m)}|n'l'\rangle|^2}{E_{nl}^{(0)} - E_{n'l'}^{(0)}} + \langle nljs|V^{(1/m^2)}|nljs\rangle. \quad (76)$$

The results for the spectrum are summarized in the tables I and II, which refer to the cases $m = 3\sqrt{\sigma}$ and $m = 10\sqrt{\sigma}$ respectively.¹⁰ The tables show all levels up to $n = 3$ and all S -wave levels up to $n = 6$. S -wave levels are degenerate in spin because the last line of (75) does not contain a spin-spin interaction. For some states the $1/m$ potential turns out to give a smaller contribution than the $1/m^2$ potentials. It happens when $\sqrt{\sigma} \langle nl|r|nl\rangle$ is close to 1, and the logarithm in the $1/m$ potential vanishes. This is the case for the $1S$ state when $m = 3\sqrt{\sigma}$: $\sqrt{\sigma} \langle 1S|r|1S\rangle \approx 1.08$, and for the $1P$ states when $m = 10\sqrt{\sigma}$: $\sqrt{\sigma} \langle 1P|r|1P\rangle \approx 1.04$. For the other states and in particular for higher states the contributions of the different potentials scale naturally. All $1/m^2$ corrections are of similar size. This holds also for the newly calculated corrections, which are listed in the column labeled V_r , showing the relevance of the spin and momentum-independent potentials.

In figure 1 we show graphically the effects of the relativistic corrections to the energy levels for the $1S$, 1^3P_J and 2^3P_J states in the cases $m = 3\sqrt{\sigma}$ and $m = 10\sqrt{\sigma}$. In figure 2 we summarize in one plot the effect of these corrections on the whole spectrum for the case $m = 3\sqrt{\sigma}$.

⁹ Kinetic energy, static potential and $E_{nl}^{(0)}$ are related by the virial theorem:

$$\langle nl|\frac{\mathbf{p}^2}{m}|nl\rangle = \frac{1}{2}\langle nl|\sigma r|nl\rangle = \frac{E_{nl}^{(0)}}{3} \sim \frac{\sigma^{2/3}}{m^{1/3}},$$

where the last relation shows the dependence of $E_{nl}^{(0)}$ on the parameters m and σ [52]. From this it follows that $1/\langle nl|r|nl\rangle \sim (\sigma m)^{1/3}$. One might therefore expect corrections of relative order σ/m^2 to be parametrically suppressed by a factor $(\sigma/m^2)^{1/3}$ with respect to corrections of relative order $1/(m\langle nl|r|nl\rangle)^2$, if $m \gg \sqrt{\sigma}$. Corrections of relative order σ/m^2 are those associated with the $1/m^2$ potentials $V_r^{(2,0)}(r)$ and $V_r^{(1,1)}(r)$. Corrections of relative order $1/(m\langle nl|r|nl\rangle)^2$ are those associated with the other $1/m^2$ potentials and with the second-order quantum-mechanical corrections induced by the $V^{(1/m)}$ potential. As we will see, however, for the range of masses considered here, the contributions to the spectrum turn out to be numerically comparable for all the $1/m^2$ potentials.

¹⁰ If $\sqrt{\sigma} = 457$ MeV [53], then $m = 3\sqrt{\sigma}$ corresponds approximately to the charm mass and $m = 10\sqrt{\sigma}$ to the bottom mass.

Levels	$E^{(0)}$	$V^{(1/m)}$	$V_{2\text{nd order}}^{(1/m)}$	$V_{\mathbf{L}^2}$	V_{LS}	V_r	E
$1S$	1.621	-0.007	-0.007	0	0	-0.021	1.586
1^1P_1	2.331	0.080	-0.005	-0.027	0	-0.030	2.349
1^3P_0	2.331	0.080	-0.005	-0.027	0.082	-0.030	2.431
1^3P_1	2.331	0.080	-0.005	-0.027	0.041	-0.030	2.390
1^3P_2	2.331	0.080	-0.005	-0.027	-0.041	-0.030	2.308
$2S$	2.834	0.100	-0.004	0	0	-0.037	2.893
1^1D_2	2.946	0.134	-0.004	-0.062	0	-0.038	2.976
1^3D_1	2.946	0.134	-0.004	-0.062	0.093	-0.038	3.069
1^3D_2	2.946	0.134	-0.004	-0.062	0.031	-0.038	3.007
1^3D_3	2.946	0.134	-0.004	-0.062	-0.062	-0.038	2.914
2^1P_1	3.387	0.147	-0.003	-0.022	0	-0.044	3.465
2^3P_0	3.387	0.147	-0.003	-0.022	0.066	-0.044	3.531
2^3P_1	3.387	0.147	-0.003	-0.022	0.033	-0.044	3.498
2^3P_2	3.387	0.147	-0.003	-0.022	-0.033	-0.044	3.432
$3S$	3.828	0.161	-0.003	0	0	-0.049	3.937
$4S$	4.706	0.203	-0.002	0	0	-0.061	4.846
$5S$	5.508	0.235	-0.002	0	0	-0.071	5.670
$6S$	6.256	0.262	-0.002	0	0	-0.081	6.435

TABLE I: Spectrum in the case $m = 3\sqrt{\sigma}$. All energies are expressed in units of $\sqrt{\sigma}$. The column $E^{(0)}$ lists the zeroth-order energy levels, which for S waves are related to the zeros of the Airy function [52]. The column $V^{(1/m)}$ lists the matrix element of $\sigma \ln(\sigma r^2)/(\pi m)$. The columns $V_{2\text{nd order}}^{(1/m)}$, $V_{\mathbf{L}^2}$, V_{LS} and V_r list the matrix elements of the second-order contribution of the $1/m$ potential and the matrix elements of $-\sigma \mathbf{L}^2/(6m^2r)$, $-\sigma \mathbf{L} \cdot \mathbf{S}/(2m^2r)$ and $-9\zeta_3 \sigma^2 r/(2\pi^3 m^2)$ respectively. The column E gives the total energy levels according to (76).

Levels	$E^{(0)}$	$V^{(1/m)}$	$V_{2\text{nd order}}^{(1/m)}$	$V_{\mathbf{L}^2}$	V_{LS}	V_r	E
$1S$	1.085	-0.028	-0.001	0	0	-0.001	1.055
1^1P_1	1.560	-0.0015	-0.0007	-0.004	0	-0.002	1.552
1^3P_0	1.560	-0.0015	-0.0007	-0.004	0.011	-0.002	1.563
1^3P_1	1.560	-0.0015	-0.0007	-0.004	0.006	-0.002	1.558
1^3P_2	1.560	-0.0015	-0.0007	-0.004	-0.006	-0.002	1.546
$2S$	1.897	0.004	-0.0005	0	0	-0.002	1.899
1^1D_2	1.972	0.015	-0.0005	-0.008	0	-0.002	1.977
1^3D_1	1.972	0.015	-0.0005	-0.008	0.013	-0.002	1.990
1^3D_2	1.972	0.015	-0.0005	-0.008	0.004	-0.002	1.981
1^3D_3	1.972	0.015	-0.0005	-0.008	-0.008	-0.002	1.969
2^1P_1	2.267	0.019	-0.0005	-0.003	0	-0.003	2.280
2^3P_0	2.267	0.019	-0.0005	-0.003	0.009	-0.003	2.289
2^3P_1	2.267	0.019	-0.0005	-0.003	0.004	-0.003	2.284
2^3P_2	2.267	0.019	-0.0005	-0.003	-0.004	-0.003	2.276
$3S$	2.562	0.023	-0.0004	0	0	-0.003	2.582
$4S$	3.150	0.035	-0.0003	0	0	-0.004	3.181
$5S$	3.687	0.045	-0.0002	0	0	-0.004	3.728
$6S$	4.188	0.053	-0.0002	0	0	-0.005	4.236

TABLE II: Spectrum in the case $m = 10\sqrt{\sigma}$, columns are like those in table I.

VI. CONCLUSIONS

The effective string theory provides an economical way to parameterize the long-range behaviour of the heavy quark-antiquark potential in the absence of available lattice data. Whenever lattice data are available they compare favourably with the EST predictions. This is the case for the static potential that has been tested also at the level of quantum fluctuations of order $1/r$, the $1/m$ potential, and the $1/m^2$ spin-orbit and momentum-dependent potentials. These successful comparisons support the assumption of a one-to-one mapping in the long range between Wilson loop expectation values and correlators of string

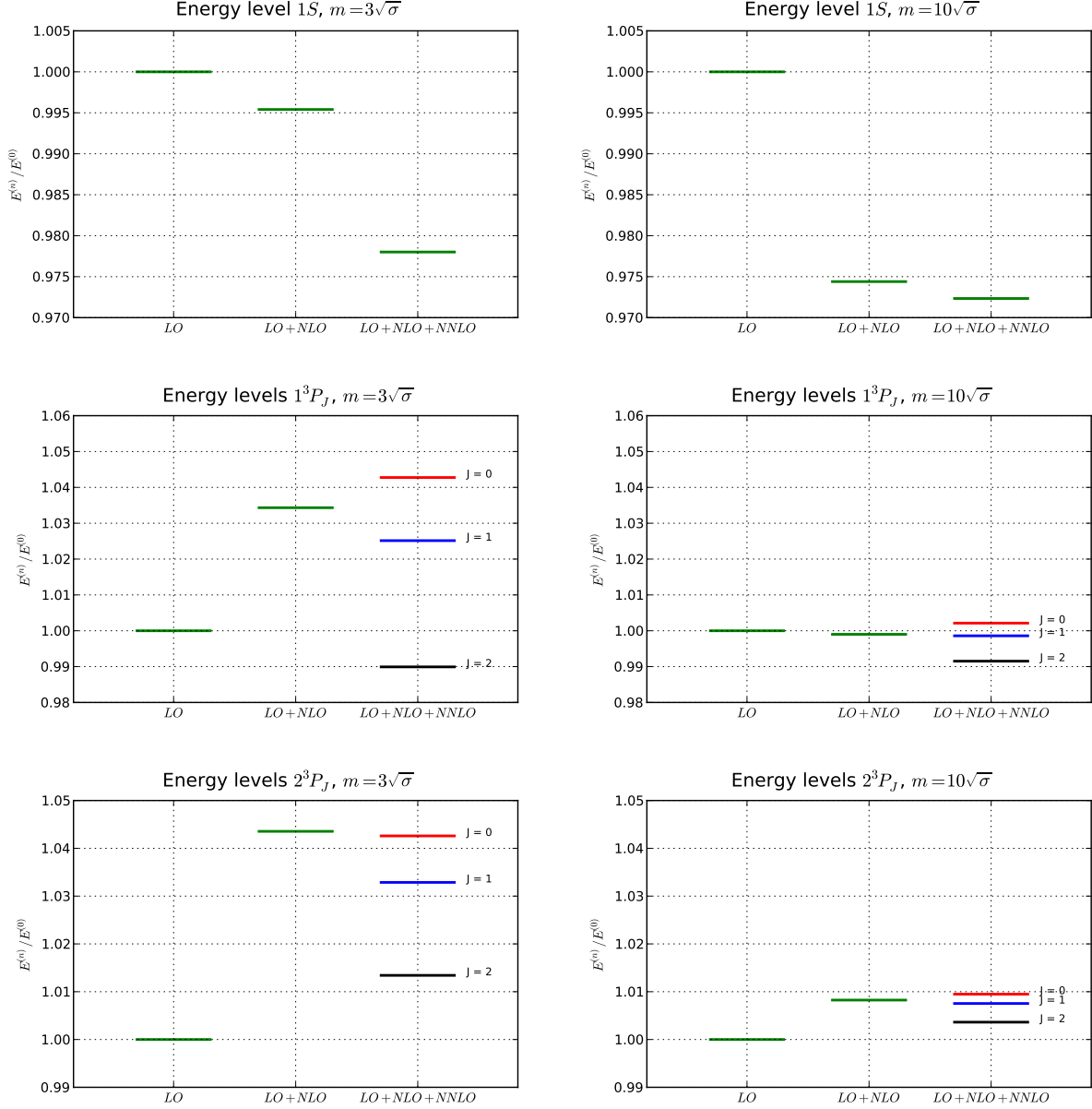


FIG. 1: Energy levels for the states $1S$, 1^3P_J and 2^3P_J normalized with respect to $E_{1S}^{(0)}$, $E_{1P}^{(0)}$ and $E_{2P}^{(0)}$ respectively. The left plots refer to the case $m = 3\sqrt{\sigma}$, the right ones to the case $m = 10\sqrt{\sigma}$. The leading order (LO) levels correspond to $E_{nl}^{(0)}$, the next-to-leading-order (NLO) corrections to $\langle nl|V^{(1/m)}|nl\rangle$ and the next-to-next-to-leading-order (NNLO) ones to the remaining two terms shown in the right-hand side of (76).

coordinates; see (36).

Existing lattice data for the spin-spin potentials are so far consistent with zero in the long range [25]. It would be interesting to produce more accurate data able to detect a

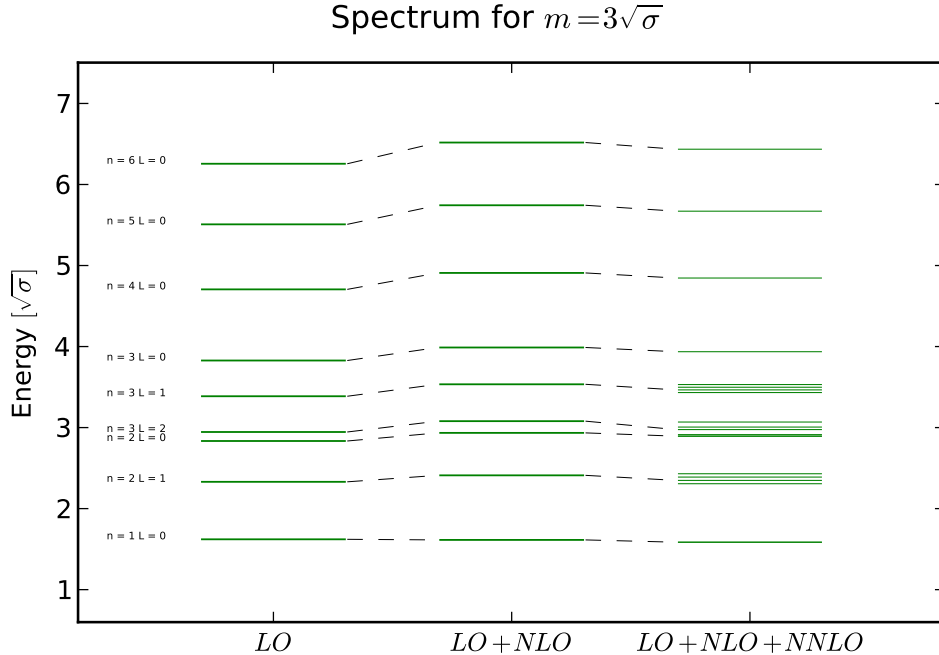


FIG. 2: Spectrum of all states up to $n = 3$ and of all S -wave states up to $n = 6$ in the case $m = 3\sqrt{\sigma}$. Energies are expressed in units of $\sqrt{\sigma}$.

long-distance signal, for the EST predicts a sharp falloff proportional to $1/r^5$.

In this paper, we have computed in the EST the momentum and spin-independent $1/m^2$ potentials. They show a linearly rising behaviour with the distance and may be interpreted as a sort of relativistic correction to the static potential. This is again a sharp prediction of the EST that can be checked against data from lattice, once calculations of Wilson loop expectation values with four chromoelectric field insertions are performed. Under the assumption of the exact mapping (36) the expressions of the potentials are given in (73) and (74). The net effect of these potentials in the equal mass case is to reduce the string tension by an amount $9\zeta_3\sigma^2/(2\pi^3m^2)$.¹¹

One may argue that the newly computed potentials are of phenomenological relevance in quarkonium physics [55, 56] since their contribution to the spectrum, at least when the short-distance part of the potentials is neglected, is comparable in size to that of the other

¹¹ It is interesting to notice that an effective reduction in the string tension due to relativistic effects may be observed in some plots of [54]. We thank Shoichi Sasaki for communications on this point.

$1/m^2$ potentials. A realistic description of quarkonium requires, however, the inclusion of the short-distance parts of the potentials. These are known from perturbation theory. Spectroscopy studies that use lattice data to parameterize the long-distance parts of the potentials and perturbation theory for the short-distance parts are for instance in [22, 53, 57, 58]. However, such studies are unavoidably incomplete insofar as not all potentials have been computed yet on the lattice. The core message of this work is that the EST may provide the missing information through the long-distance expression of the potential. In the model defined by equation (75), that expression depends on just two parameters: the heavy-quark mass and the string tension. It therefore provides a simple infrared completion of the heavy quark-antiquark potential valuable for future quarkonium studies [59].

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